Applied Mathematical Sciences, Vol. 9, 2015, no. 55, 2729 - 2748
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2015.52158

A New Generalized of Transmuted Lindley Distribution

Mahmoud M. Mansour

Department of Statistics, Mathematics and Insurance
Benha University, Egypt

Salah M. Mohamed

Institute of Statistical Studies & Research
Cairo University, Egypt

Abstract

This paper introduces a new generalization of the transmuted Lindley distribution, based on a new family of life time distribution by Mansour et al. (2015). We refer to the new distribution as transmuted Lindley distribution (NTL) distribution. The new model contains some of lifetime distributions as special cases such as exponentiated Lindley, transmuted Lindley and Lindley distributions. The properties of the new model are discussed and the maximum likelihood estimation is used to evaluate the parameters. Explicit expressions are derived for the moments and examine the order statistics. It will be shown that the analytical results are applicable to model real data.

Keywords: transmutation; survival function; exponentiated exponential; order statistics; maximum likelihood estimation

1. Introduction

The quality of the procedures used in a statistical analysis depends heavily on the assumed probability model or distributions. Because of it, considerable effort has been expended in the development of large classes of standard probability distributions along with relevant statistical methodologies. However, there still remain many important problems where the real data does not follow any of the classical or standard probability models. Since there is a clear need for extended forms of these distributions a significant progress has been made toward the generalization of some well known distributions and their successful application to problems in areas such as engineering, finance, economics and biomedical sciences, among others.

A lot of distributions have been made using cumulative distribution function (cdf)$G(x)$, probability density function (pdf)$g(x)$, or survival function $\bar{G}(x)$ that one can rely on, as a baseline distribution, to introduce new models. The Exponentiated generalization is the first generalization allowing for non-monotone hazard rates, including the bathtub shaped hazard rate. The cdf of the new distribution is defined by $F(x) = G^{\propto}(x)$, where $\propto > 0$. The exponentiated exponential distribution has been introduced by Ahuja and Nash (1967), and further studied by Gupta and Kundu (1999). The first generalization allowing for nonmonotone hazard rates, including the bathtub shaped hazard rate, is the exponentiated Weibull (EW) distribution due to Mudholkar and Srivastava (1993), and Mudholkar et al. (1995).

An interesting idea of generalizing a distribution, known in the literature by transmutation, is derived by using the Quadratic Rank Transmutation Map (QRTM) introduced by Shaw and Buckley (2009). Merovci (2013) introduced transmuted Lindley distribution. According to the transmutation generalization approach, the cdf satisfies the relationship

$$F(x) = (1+\lambda)G(x) - \lambda[G(x)]^2, \tag{1}$$

where G(x) the cdf of the baseline distribution.

Mansour et al. (2015) introduced new transmutation map approach to define a new model which Lindley distribution. The proposed modification generalizes the rank of the transmutation map by replacing the constant power by additional parameters. The following definition gives the mechanism of generating a new family of lifetime distributions building on a base model, that is, according to this modification.

Definition 1 Let $G(x)$ be the cumulative distribution function (cdf) of a non-negative absolutely continuous random variable, $G(x)$ be strictly increasing on its support, and $G(0) = 0$ define a new cdf, F(x), out of $G(x)$ as

$$F(x) = (1+\lambda)[G(x)]^{\delta} - \lambda[G(x)]^{\alpha}, x > 0, \tag{2}$$

where $\alpha, \delta > 0$ for $0 > \lambda > -1$ and $\alpha > 0$, $(\alpha + \alpha/4) \geq \delta \geq (\alpha/2)$ for $0 < \lambda < 1$.

The rest of the article is organized as follows. In Section2, introduces the proposed new transmuted Lindley model according to the new class of distribution. In Section 3, we find the reliability function, hazard rate and cumulative hazard rate of the subject model. The Expansion for the pdf and the cdf Functions is derived in Section 4. In section 5, The statistical properties include quantile functions, median , moments , and moment generating function are given,. In Section 6, order statistics are discussed. In Section 7, we introduce the method of likelihood estimation as point estimation and, give the equation used to estimate the parameters, using the maximum product spacing estimates and the least square estimates techniques. Finally, we fit the distribution to real data set to examine it.

2. A New Transmuted Lindley Distribution

Definition 2.1 A random variable X is said to have the Lindley distribution with parameter θ if its probability density is defined as

$$f(x) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}, x > 0, \theta > 0. \tag{3}$$

The corresponding cumulative distribution function (cdf) is:

$$F(x) = 1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}, x > 0, \theta > 0. \tag{4}$$

Now using (2) and (4) we have the cdf of a new transmuted Lindley distribution

$$F(x) = (1+\lambda)\left[1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^{\delta} - \lambda\left[1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^{\alpha}, x > 0, \tag{5}$$

Hence, the pdf of new transmuted Lindley distribution is

$$f(x) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\left((1+\lambda)\delta\left[1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^{\delta-1} - \lambda\alpha\left[1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^{\alpha-1}\right), \tag{6}$$

where $\theta, \alpha, \delta > 0$, for $0 > \lambda > -1$ and,

$\theta, \alpha > 0$, $(\alpha + \alpha/4) \geq \delta \geq (\alpha/2)$ for $0 < \lambda < 1$.

We present special cases of the new transmuted Lindley distribution (NTLD) as follows:

Transmuted Lindley distribution: for $\alpha = 2 \; and \; \delta = 1$, the distribution function (5) becomes

$$F(x) = (1+\lambda)\left[1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right] - \lambda\left[1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^{2}, x > 0, \tag{7}$$

which is the distribution function of the transmuted Lindley distribution.

Transmuted exponentiated Lindley distribution: for $\delta = \frac{\alpha}{2}$, the distribution function (5) becomes

$$F(x) = (1+\lambda)\left[1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^{\frac{\alpha}{2}} - \lambda\left[1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^{\alpha}, x > 0, \quad (8)$$

which is the distribution function of the transmuted exponentiated Lindley distribution.

Exponentiated Lindley distribution: for $\lambda = 0$, the distribution function (5) becomes

$$F(x) = \left[1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^{\delta}, x > 0, \quad (9)$$

which is the distribution function of the exponentiated Lindley distribution.

Lindley distribution: for $\lambda = 0, \delta = 1$, the distribution function (5) becomes

$$F(x) = \left[1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right], x > 0, \quad (10)$$

which is the distribution function of the Lindley distribution.

Figures 1 and 2 illustrates some of the possible shapes of the pdf and cdf of the NTL distribution for selected values of the parameters λ, θ, δ *and* αrespectively

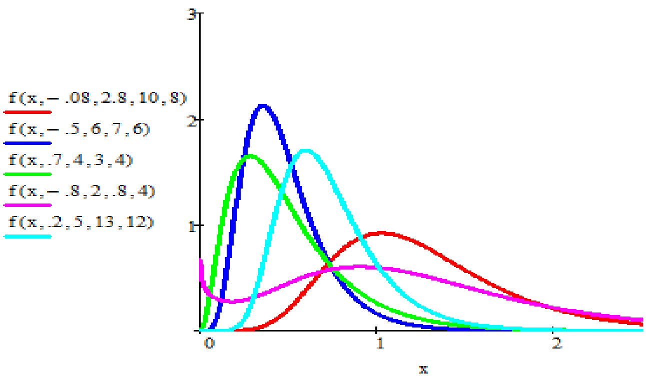

Figure 1: Probability density function of the NTL distribution.

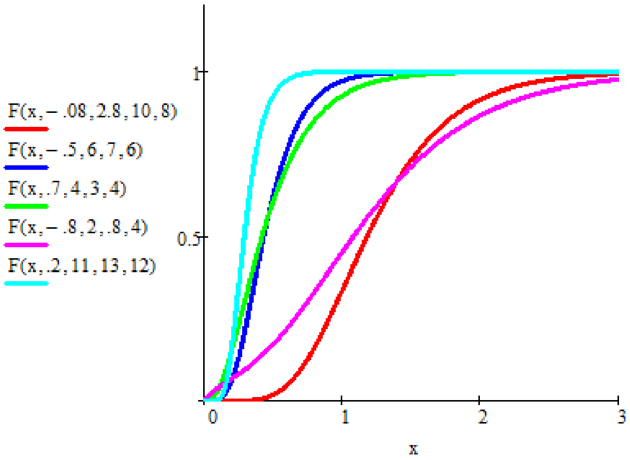

Figure 2: Distributionf of the NTL distribution.

3. Reliability Analysis

The characteristics in reliability analysis which are the reliability function (RF), the hazard rate function (HF) and the cumulative hazard rate function (CHF) for the NTL are introduces in this section.

3.1 Reliability Function

The reliability function (RF) also known as the survival function, which is the probability of an item not failing prior to some time t, is defined by $R(x) = 1 - F(x)$. The reliability function of the new transmuted Lindley distribution (NTLD) denoted by $R_{NTL}(\lambda, \theta, \delta, \alpha)$, can be a useful characterization of life time data analysis. It can be defined as,

$$R_{NTL}(\lambda, \theta, \delta, \alpha) = 1 - F_{NTL}(\lambda, \theta, \delta, \alpha),$$

the survival function of is given by,

$$R_{NTL}(x, \lambda, \theta, \delta, \alpha) = 1 - \left[(1+\lambda)\left[1 - \frac{\theta + 1 + \theta x}{\theta + 1} e^{-\theta x}\right]^{\delta} - \lambda\left[1 - \frac{\theta + 1 + \theta x}{\theta + 1} e^{-\theta x}\right]^{\alpha}\right]. \tag{11}$$

Figure 3 illustrates the pattern of the called the new transmuted Lindley distribution reliability function with different choices of parameters λ, θ, δ *and* α.

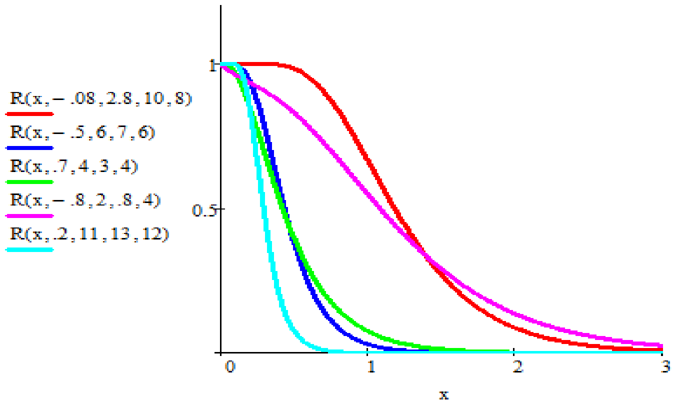

Figure 3: Reliability function of the NTL distribution.

3.2 Hazard Rate Function

The other characteristic of interest of a random variable is the hazard rate function (HF). the new transmuted Lindley distribution also known as instantaneous failure rate denoted by $h_{NTL}(x)$, is an important quantity characterizing life phenomenon. It can be loosely interpreted as the conditional probability of failure, given it has survived to the time t. The HF of the NTL is defined by $h_{NTL}(x,,\lambda,\theta,\delta,\alpha) = f_{NTL}(x,,\lambda,\theta,\delta,\alpha)/R_{NTL}(x,,\lambda,\theta,\delta,\alpha)$,

Figure 4 illustrates some of the possible shapes of the hazard rate function of the new transmuted Lindley distribution for different values of the parameters $\lambda, \beta, a, b, \delta$and α.

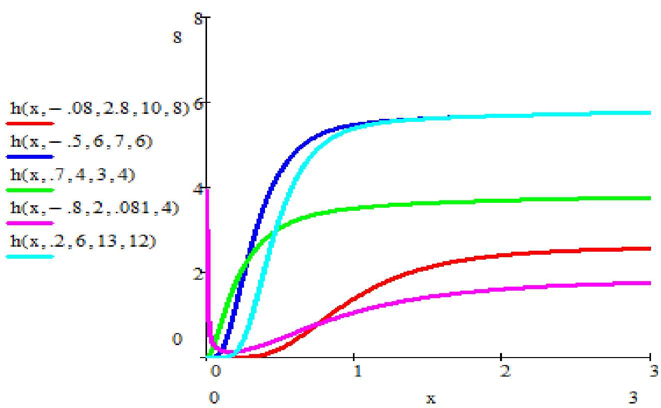

Figure 4: Hazard rate of the NTL distribution.

3.3 Cumulative Hazard Rate Function

The Cumulative hazard function (CHF) of the new transmuted Lindley distribution, denoted by $H_{NTL}(x, \lambda, \theta, \delta, \alpha)$, is defined as $H_{NTL}(x, \lambda, \theta, \delta, \alpha) = \int_0^x h_{NTL}(x, \lambda, \theta, \delta, \alpha)dx = -\ln R_{NTL}(x, \lambda, \theta, \delta, \alpha)$,

$$H_{NTL}(x, \lambda, \theta, \delta, \alpha) = -\ln\left(1 - \left[(1+\lambda)\left[1 - \frac{\theta + 1 + \theta x}{\theta + 1}e^{-\theta x}\right]^{\delta} - \lambda\left[1 - \frac{\theta + 1 + \theta x}{\theta + 1}e^{-\theta x}\right]^{\alpha}\right]\right). \qquad (12)$$

4. Expansion for the pdf and the cdf Functions

In this section we introduced another expression for the pdf and the cdf functions using. The Maclaurin expansion to simplifying the pdf and the cdf forms.

4.1 Expansion for the pdf Function

From equation (6) and using the expansions

$$(1-z)^k = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(k+1)}{\Gamma(k-j+1)j!} z^j. \qquad (13)$$

Which holds for $|z| < 1$ and $k > 0$.

and

$$(a-b)^k = \sum_{j=0}^{k} \binom{k}{j} (-1)^j b^j a^{k-j}. \qquad (14)$$

Using (14) and applying it to (6), the pdf of the NTL model can be written as:

$$f(x) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x} \sum_{i=0}^{1} (-1)^i \lambda^i \alpha^i \left[1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^{i(\alpha-1)+(\delta-1)(1-i)} \times (1+\lambda)^{1-i}\delta^{1-i}, \qquad (15)$$

using (14) and applying it to (15), the pdf of the NTL model can be written as:

$$f(x) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x} \sum_{i=0}^{1}\sum_{j=0}^{\infty} \frac{(-1)^{i+j}\Gamma(i(\alpha-1)+(\delta-1)(1-i)+1)}{j!\,\Gamma(i(\alpha-1)+(\delta-1)(1-i)-j+1)} \times \lambda^i \alpha^i (1+\lambda)^{1-i}\delta^{1-i} \left(\frac{\theta+1+\theta x}{\theta+1}\right)^j e^{-\theta j x}, \qquad (16)$$

the pdf of the NTL model can be written as:

$$f(x) = \frac{\theta^2}{\theta+1} \sum_{i=0}^{1}\sum_{j=0}^{\infty}\sum_{k=0}^{j} \frac{(-1)^{i+j}\Gamma(i(\alpha-1)+(\delta-1)(1-i)+1)\theta^k}{k!\,\Gamma(j-k+1)\Gamma(i(\alpha-1)+(\delta-1)(1-i)-j+1)(\theta+1)^j} \times \left(\times \lambda^i \alpha^i (1+\lambda)^{1-i}\delta^{1-i}\,(1+x)^{k+1}\, e^{-\theta x(j+1)}\right), \qquad (17)$$

$$f(x)=\frac{\theta^2}{\theta+1}\sum_{i=0}^{1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{k+1}\frac{(-1)^{i+j}\Gamma(i(\alpha-1)+(\delta-1)(1-i)+1)k\theta^k}{l!\,\Gamma(k-l+2)\Gamma(j-k+1)\Gamma(i(\alpha-1)+(\delta-1)(1-i)-j+1)(\theta+1)^j} \times\left(\lambda^i\alpha^i(1+\lambda)^{1-i}\delta^{1-i}\ x^l\ e^{-\theta x(j+1)}\right), \tag{18}$$

the pdf of NTL distribution can then be represented as:

$$f(x)=\sum_{i=0}^{1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{k+1}A_{i:l}\times\left(x^l\ e^{-\theta x(j+1)}\right), \tag{19}$$

where $A_{i:l}$ is a constant term given by,

$$A_{i:l}=\frac{\theta^2}{\theta+1}\times\frac{(-1)^{i+j}\Gamma(i(\alpha-1)+(\delta-1)(1-i)+1)k\theta^k\lambda^i\alpha^i(1+\lambda)^{1-i}\delta^{1-i}}{l!\,\Gamma(k-l+2)\Gamma(j-k+1)\Gamma(i(\alpha-1)+(\delta-1)(1-i)-j+1)(\theta+1)^j}.$$

4.2 Expansion for the cdf Function

Using expansion (13) and (14) to Equation (5) ,then the cdf function of the NTL can be written as:

$$F(x)=\sum_{i=0}^{1}\sum_{j,m=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{k}B_{i:l}x^{l+m}, \tag{20}$$

where $B_{i:l}$ is a constant term given by:

$$B_{i:l}=\frac{(-1)^{i+j+m}\lambda^i(1-\lambda)^{1-i}\Gamma(\alpha i+\delta(1-i)+1)\theta^{k+m}j^m}{l!\,m!\,\Gamma(\alpha i+\delta(1-i)-j+1)\Gamma(j-k+1)\Gamma(k-l+1)(1+\theta)^j}.$$

5. Statistical properties

In this section we discuss the most important statistical properties of the NTL distribution.

5.1 Quantile function

The quantile function is obtained by inverting the cumulative distribution (20), where the p-th quantile x_p of the NTL model is the real solution of the following equation:

$$\sum_{i=0}^{1}\sum_{j,m=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{k}B_{i:l}{x_p}^{l+m}-p=0.$$

An expansion for the median M follows by taking $p=0.5$.

5.2 Moments

The r^{th} non-central moments or (moments about the origin) of the NTL under using equation (19) is given by:

$$\mu_r^{'} = E(X^r) = \int_0^\infty X^r f(x)dx,$$

$$\mu_r^{'} = \int_0^\infty X^r \left[\sum_{i=0}^{1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{k+1} A_{i:l}\left(x^l\ e^{-\theta x(j+1)}\right)\right]dx,$$

then,

$$\mu_r^{'} = \sum_{i=0}^{1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{k+1} \frac{A_{i:l}\Gamma(r+l+1)}{(\theta(j+1))^{r+l+1}}. \tag{21}$$

In particular, when r = 1, Eq. (21) yields the mean of the NTL distribution, μ as

$$\mu = \sum_{i=0}^{1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{k+1} \frac{A_{i:l}\Gamma(l+2)}{(\theta(j+1))^{l+2}}.$$

The n^{th} central moments or (moments about the mean) can be obtained easily from the r^{th} non-central moments throw the relation:

$$m_u = E(X-\mu)^n = \sum_{r=0}^{n}(-\mu)^{n-r}E(X^r).$$

Then the n^{th} central moments of the NTL is given by:

$$m_u = E(X-\mu)^n = \sum_{r=0}^{n}(-\mu)^{n-r}\sum_{i=0}^{1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{k+1} \frac{A_{i:l}\Gamma(r+l+1)}{(\theta(j+1))^{r+l+1}}.$$

5.3 The Moment Generating Function

The moment generating function, $M_x(t)$, can be easily obtained from the r^{th} non-central moment through the relation

$$M_x(t) = \int_0^\infty e^{tx} f(x)dx,$$

$$M_x(t) = \int_{0}^{\infty} \mathrm{e}^{\mathrm{tx}} \left[\sum_{i=0}^{1} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \sum_{l=0}^{k+1} A_{i:l} \left(x^l \; e^{-\theta x(j+1)} \right) \right] \mathrm{dx},$$

$M_x(t)$ =Then, the moment generating function of the NTL distribution is given by,

$$M_x(t) = \sum_{i=0}^{1} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \sum_{l=0}^{k+1} \frac{A_{i:l}\Gamma(l+1)}{(\theta(j+1)+t)^{l+1}}.$$

6. Order Statistics

Let $X_1, X_2, \dots, X_n$ denote n-independent random variables from a distribution function $F_X(x)$ with pdf $f_X(x)$. Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be the ordered sample arrangement. The pdf of $X_{(j)}$ is given by:

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!\,(n-j)!} f_X(x)[F_X(x)]^{j-1}.[1-F_X(x)]^{n-j}, \quad j = 1,2,\dots,n.$$

Then from (5) and (6) the pdf of $X_{(j)}$ is given by:

$$f(x) = \frac{n!}{(j-1)!\,(n-j)!} \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\left((1+\lambda)\delta[\mathrm{I}(x,\theta)]^{\delta-1} - \lambda\alpha[\mathrm{I}(x,\theta)]^{\alpha-1}\right)$$
$$\times \left[(1+\lambda)[\mathrm{I}(x,\theta)]^{\delta} - \lambda[\mathrm{I}(x,\theta)]^{\alpha}\right]^{j-1}$$
$$\times \left[1 - \left((1+\lambda)[\mathrm{I}(x,\theta)]^{\delta} - \lambda[\mathrm{I}(x,\theta)]^{\alpha}\right)\right]^{n-j}.$$

Where $\mathrm{I}(x,\theta) = 1 - \frac{\theta+1+\theta x}{\theta+1} e^{-\theta x}$.

Therefore, the pdfs of the smallest and the largest order statistic are respectively given by:

$$f_{X_{(1)}}(x) = n\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\left((1+\lambda)\delta[\mathrm{I}(x,\theta)]^{\delta-1} - \lambda\alpha[\mathrm{I}(x,\theta)]^{\alpha-1}\right)$$
$$\times \left[1 - \left((1+\lambda)[\mathrm{I}(x,\theta)]^{\delta} - \lambda[\mathrm{I}(x,\theta)]^{\alpha}\right)\right]^{n-1},$$

and

$$f_{X_{(n)}}(x) = n\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\left((1+\lambda)\delta[\mathrm{I}(x,\theta)]^{\delta-1} - \lambda\alpha[\mathrm{I}(x,\theta)]^{\alpha-1}\right)$$
$$\times \left[(1+\lambda)[\mathrm{I}(x,\theta)]^{\delta} - \lambda[\mathrm{I}(x,\theta)]^{\alpha}\right]^{n-1}.$$

7. Estimation of the Parameters

In this section we introduce the method of likelihood to estimate the parameters involved, then gives the equation used to estimate the parameters using the maximum product spacing estimates and the least square estimates techniques.

7.1 Maximum Likelihood Estimation

The maximum likelihood estimators (MLEs) for the parameters of the new transmuted Lindely distribution NTL $(\lambda, \theta, \alpha, \delta)$ is discussed in this section. Consider the random sample $x_1, x_2, \ldots, x_n$ of size n from NTL$(\lambda, \theta, \alpha, \delta)$ with probability density function in (6), then the likelihood function can be expressed as follows

$$\mathrm{L}(x_1, x_2, \ldots, x_n, \lambda, \theta, \alpha, \delta) = \prod_{i=1}^{n} f_{\mathrm{NTL}}(x_i, \lambda, \theta, \alpha, \delta),$$

$$\mathrm{L}(x_1, x_2, \ldots, x_n, \lambda, \theta, \alpha, \delta) = \prod_{i=1}^{n} \frac{\theta^2}{\theta+1}(1+x_i)e^{-\theta x_i}\left((1+\lambda)\delta[\mathrm{I}(x_i,\theta)]^{\delta-1} - \lambda\alpha[\mathrm{I}(x_i,\theta)]^{\alpha-1}\right)$$

Hence, the log-likelihood function τ= ln L becomes,

$$\tau = 2\mathrm{n}\ln\theta - n\ln(\theta+1) + \sum_{i=}^{n}\ln(1+x_i) - \sum_{i=1}^{n}\theta x_i + \sum_{i=1}^{n}\ln\left[(1+\lambda)\delta[\mathrm{I}(x_i,\theta)]^{\delta-1} - \lambda\alpha[\mathrm{I}(x_i,\theta)]^{\alpha-1}\right] \tag{22}$$

Differentiating Equation (22) with respect to λ, θ, δ and α then equating it to zero, we obtain the MLEs of λ, θ, δ and α as follows,

$$\frac{\partial\tau}{\partial\lambda} = \sum_{i=1}^{n} \frac{\delta[\mathrm{I}(x_i,\theta)]^{\delta-1} - \alpha[\mathrm{I}(x_i,\theta)]^{\alpha-1}}{(1+\lambda)\delta[\mathrm{I}(x_i,\theta)]^{\delta-1} - \lambda\alpha[\mathrm{I}(x_i,\theta)]^{\alpha-1}}, \tag{23}$$

$$\frac{\partial\tau}{\partial\theta} = \frac{2n}{\theta} - \frac{n}{\theta+1} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i e^{-\theta x_i}\left[\left(1+\frac{\theta x_i}{\theta+1}\right) - \left(\frac{1}{(\theta+1)^2}\right)\right] \times \frac{\left(\delta(\delta-1)(1+\lambda)[\mathrm{I}(x_i,\theta)]^{\delta-2} - \alpha(\alpha-1)\lambda[\mathrm{I}(x_i,\theta)]^{\alpha-2}\right)}{(1+\lambda)\delta[\mathrm{I}(x_i,\theta)]^{\delta-1} - \lambda\alpha[\mathrm{I}(x_i,\theta)]^{\alpha-1}}, \tag{24}$$

$$\frac{\partial\tau}{\partial\delta} = \sum_{i=1}^{n} \frac{(1+\lambda)[\mathrm{I}(x_i,\theta)]^{\delta-1}\left(\delta\ln\left(\mathrm{I}(x_i,\theta)\right)+1\right)}{(1+\lambda)\delta[\mathrm{I}(x_i,\theta)]^{\delta-1} - \lambda\alpha[\mathrm{I}(x_i,\theta)]^{\alpha-1}}, \tag{25}$$

and

$$\frac{\partial\tau}{\partial\alpha} = \sum_{i=1}^{n} \frac{-\lambda[\mathrm{I}(x_i,\theta)]^{\alpha-1}\left(\alpha\ln\left(\mathrm{I}(x_i,\theta)\right)+1\right)}{(1+\lambda)\delta[\mathrm{I}(x_i,\theta)]^{\delta-1} - \lambda\alpha[\mathrm{I}(x_i,\theta)]^{\alpha-1}}, \tag{26}$$

We can find the estimates of the unknown parameters by maximum likelihood method by setting these above nonlinear system of Equations (23) - (26) to zero and solve them simultaneously. These solutions will yield the ML estimators $\hat{\lambda}, \hat{\theta}, \hat{\delta}$ and $\hat{\alpha}$. For the four parameters new transmuted Lindely distribution NTL$(x, \lambda, \theta, \alpha, \delta) pdf$ all the second order derivatives exist. Thus we have the inverse dispersion matrix is given by

$$\begin{pmatrix} \hat{\lambda} \\ \hat{\theta} \\ \hat{\delta} \\ \hat{\alpha} \end{pmatrix} \sim N \left[\begin{pmatrix} \lambda \\ \theta \\ \delta \\ \alpha \end{pmatrix}, \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} & \hat{V}_{14} \\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} & \hat{V}_{24} \\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} & \hat{V}_{34} \\ \hat{V}_{41} & \hat{V}_{42} & \hat{V}_{43} & \hat{V}_{44} \end{pmatrix} \right],$$

$$V^{-1} = -E \begin{pmatrix} V_{11} & V_{12} & V_{13} & V_{14} \\ V_{21} & V_{22} & V_{23} & V_{24} \\ V_{31} & V_{32} & V_{33} & V_{34} \\ V_{41} & V_{42} & V_{43} & V_{44} \end{pmatrix}. \tag{27}$$

Equation (27) is the variance covariance matrix of the NTL$(x, \lambda, \theta, \alpha, \delta)$ where

$$V_{11} = \frac{\partial^2 \Psi}{\partial \lambda^2} \qquad V_{12} = \frac{\partial^2 \Psi}{\partial \lambda \, \partial \theta} \qquad V_{13} = \frac{\partial^2 \Psi}{\partial \lambda \, \partial \delta} \qquad V_{14} = \frac{\partial^2 \Psi}{\partial \lambda \, \partial \alpha}$$

$$V_{22} = \frac{\partial^2 \Psi}{\partial \theta^2} \qquad V_{23} = \frac{\partial^2 \Psi}{\partial \theta \, \partial \delta} \qquad V_{24} = \frac{\partial^2 \Psi}{\partial \theta \, \partial \alpha}$$

$$V_{33} = \frac{\partial^2 \Psi}{\partial \delta^2} \qquad V_{34} = \frac{\partial^2 \Psi}{\partial \delta \, \partial \alpha}$$

$$V_{44} = \frac{\partial^2 \Psi}{\partial \delta^2}$$

By solving this inverse dispersion matrix, these solutions will yield the asymptotic variance and covariances of these MLEs for $\hat{\lambda}, \hat{\theta}, \hat{\delta}$ and $\hat{\alpha}$. Approximate $100(1-\phi)\%$ confidence intervals for λ, θ, δ and α can be determined as:

$$\hat{\lambda} \pm Z_{\frac{\phi}{2}} \sqrt{\hat{V}_{11}},\ \hat{\theta} \pm Z_{\frac{\phi}{2}} \sqrt{\hat{V}_{22}},\ \hat{\delta} \pm Z_{\frac{\phi}{2}} \sqrt{\hat{V}_{33}} \text{ and } \hat{\alpha} \pm Z_{\frac{\phi}{2}} \sqrt{\hat{V}_{44}},$$

where $Z_{\frac{\phi}{2}}$ is the upper ϕth percentile of the standard normal distribution.

7.2 Maximum product spacing estimates

The maximum product spacing (MPS) method has been proposed by Cheng and Amin (1983). This method is based on an idea that the differences (Spacing) of the consecutive points should be identically distributed. The geometric mean of the differences is given as

$$GM = \sqrt[n+1]{\prod_{i=1}^{n+1} D_i}, \tag{28}$$

where, the difference D_i is defined as

$$D_i = \int_{x_{(i-1)}}^{x_{(i)}} f(x,\lambda,\theta,\delta,\alpha)dx\,; \quad i = 1,2,\dots,n+1, \tag{29}$$

where, $F\left(x_{(0)},\lambda,\theta,\delta,\alpha\right) = 0$ and $F\left(x_{(n+1)},\lambda,\theta,\delta,\alpha\right) = 0$. The MPS estimators $\hat{\lambda}_{PS}$, $\hat{\theta}_{PS}$, $\hat{\alpha}_{PS}$ and $\hat{\delta}_{PS}$ of λ,θ,α and δ are obtained by maximizing the geometric mean (*GM*) of the differences. Substituting pdf of NTL distribution in (29) and taking logarithm of the above expression, we will have

$$\log GM = \frac{1}{n+1}\sum_{i=1}^{n+1} \log\left[F\left(x_{(i)},\lambda,\theta,\delta,\alpha\right) - F\left(x_{(i-1)},\lambda,\theta,\delta,\alpha\right)\right]. \tag{30}$$

The MPS estimators $\hat{\lambda}_{PS}$, $\hat{\theta}_{PS}$, $\hat{\delta}_{PS}$ and $\hat{\alpha}_{PS}$ of λ,θ,δ and α can be obtained as the simultaneous solution of the following non-linear equations:

$$\frac{\partial\, log\, GM}{\partial\lambda} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F_{\lambda}^{'}\left(x_{(i)},\lambda,\theta,\delta,\alpha\right) - F_{\lambda}^{'}\left(x_{(i-1)},\lambda,\theta,\delta,\alpha\right)}{F\left(x_{(i)},\lambda,\theta,\delta,\alpha\right) - F\left(x_{(i-1)},\lambda,\theta,\delta,\alpha\right)}\right] = 0,$$

$$\frac{\partial\, log\, GM}{\partial\theta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F_{\theta}^{'}\left(x_{(i)},\lambda,\theta,\delta,\alpha\right) - F_{\theta}^{'}\left(x_{(i-1)},\lambda,\theta,\delta,\alpha\right)}{F\left(x_{(i)},\lambda,\theta,\delta,\alpha\right) - F\left(x_{(i-1)},\lambda,\theta,\delta,\alpha\right)}\right] = 0,$$

$$\frac{\partial\, log\, GM}{\partial\delta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F_{\delta}^{'}\left(x_{(i)},\lambda,\theta,\delta,\alpha\right) - F_{\delta}^{'}\left(x_{(i-1)},\lambda,\theta,\delta,\alpha\right)}{F\left(x_{(i)},\lambda,\theta,\delta,\alpha\right) - F\left(x_{(i-1)},\lambda,\theta,\delta,\alpha\right)}\right] = 0,$$

and

$$\frac{\partial\, log\, GM}{\partial\alpha} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F_{\alpha}^{'}\left(x_{(i)},\lambda,\theta,\delta,\alpha\right) - F_{\alpha}^{'}\left(x_{(i-1)},\lambda,\theta,\delta,\alpha\right)}{F\left(x_{(i)},\lambda,\theta,\delta,\alpha\right) - F\left(x_{(i-1)},\lambda,\theta,\delta,\alpha\right)}\right] = 0.$$

7.3 Least square estimates

Let $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ be the ordered sample of size n drawn the NTL distribution. Then, the expectation of the empirical cumulative distribution function is defined as

$$E\left[F\left(X_{(i)}\right)\right] = \frac{i}{n+1}; \quad i = 1,2,\dots,n. \tag{31}$$

The least square estimates $\hat{\lambda}_{LS}$, $\hat{\theta}_{LS}$, $\widehat{\delta}_{LS}$ and $\widehat{\alpha}_{LS}$ of λ, θ, δ and $\alpha$are obtained by minimizing

$$Z(\lambda,\theta,\delta,\alpha) = \sum_{i=1}^{n}\left[F\left(x_{(i)},\lambda,\theta,\delta,\alpha\right) - \frac{i}{n+1}\right]^2.$$

Therefore, $\hat{\lambda}_{LS}$, $\hat{\theta}_{LS}$, $\widehat{\delta}_{LS}$ and $\widehat{\alpha}_{LS}$ of λ, θ, δ and α can be obtained as the solution of the following system of equations:

$$\frac{\partial Z(\lambda,\theta,\delta,\alpha)}{\partial\lambda} = \sum_{i=1}^{n} F_{\lambda}^{'}\left(x_{(i)},\lambda,\theta,\delta,\alpha\right)\left(F\left(x_{(i)},\lambda,\theta,\delta,\alpha\right) - \frac{i}{n+1}\right) = 0,$$

$$\frac{\partial Z(\lambda,\theta,\delta,\alpha)}{\partial\theta} = \sum_{i=1}^{n} F_{\theta}^{'}\left(x_{(i)},\lambda,\theta,\delta,\alpha\right)\left(F\left(x_{(i)},\lambda,\theta,\delta,\alpha\right) - \frac{i}{n+1}\right) = 0,$$

$$\frac{\partial Z(\lambda,\theta,\delta,\alpha)}{\partial\delta} = \sum_{i=1}^{n} F_{\delta}^{'}\left(x_{(i)},\lambda,\theta,\delta,\alpha\right)\left(F\left(x_{(i)},\lambda,\theta,\delta,\alpha\right) - \frac{i}{n+1}\right) = 0,$$

and

$$\frac{\partial Z(\lambda,\theta,\delta,\alpha)}{\partial\alpha} = \sum_{i=1}^{n} F_{\alpha}^{'}\left(x_{(i)},\lambda,\theta,\delta,\alpha\right)\left(F\left(x_{(i)},\lambda,\theta,\delta,\alpha\right) - \frac{i}{n+1}\right) = 0,$$

These non-linear can be routinely solved using Newton's method or fixed point iteration techniques. The subroutines to solve non-linear optimization problem are available in R, software namely optim( ), nlm( ) and bbmle( ) etc. We used nlm ( ) package for optimizing (22).

8. Application

In this section, we use a real data set to show that the new transmuted Lindley distribution can be a better model than one based on the Lindley distribution. The data set represents an uncensored data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients reported in Merovci (2013). Some summary statistics for the data are as follows:

Min	1st Qu	Median	Mean	3rd Qu	Max.
0.080	**3.348**	**6.395**	**9.366**	**11.840**	**79.05**

In order to compare the two distribution models, we consider criteria like KS (Kolmogorov Smirnov), $-2\mathcal{L}$, AIC (Akaike information criterion), and AICC (corrected Akaike information criterion) for the data set. The better distribution corresponds to smaller KS, $-2\mathcal{L}$, AIC and AICC values:

$$\mathrm{AIC} = -2\mathcal{L} + 2\mathrm{k},$$

and

$$\mathrm{AIC_C} = -2\mathcal{L} + \left(\frac{2\mathrm{kn}}{\mathrm{n}-\mathrm{k}-1}\right),$$

where $\mathcal{L}$ denotes the log-likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters, and n is the sample size.

Also, for calculating the values of KS we use the sample estimates of , λ, θ , α and δ. Table 1 shows the parameter estimation based on the maximum likelihood and least square estimation, and gives the values of the criteria AIC, AICC and KS test. The values in Table 1 indicate that the NTL distribution is a strong competitor to other distributions used here for fitting data.

Table 1. MLEs the measures AIC, AICC and KS test to data for the models.

Model	*Parameter Estimates*	Standard Error	-2LL	AIC	$\mathrm{AIC_C}$	KS
New Transmuted Lindley	$\lambda = -0.0351$	0.01989	412.854	833.7082	834.0334	0.060528
	$\theta = 0.20438$	0.0229				
	$\delta = 0.85950$	0.11557				
	$\alpha = 380.478$	332.62				
Transmuted Lindley	$\lambda = 0.61687$	0.1688	415.155	834.3101	834.4061	0.226523
	$\theta = 0.1557$	0.0150				
Exponentiated Lindley	$\alpha = 0.1648$	0.01664	416.285	836.5719	836.6679	0.092791
	$\theta = 0.733$	0.0912				
Lindley	$\theta = 0.1960$	0.01234	419.529	841.0598	841.0916	0.116398
Weighted Lindley	$\alpha = 0.15945$	0.0172	416.442	836.8845	836.9805	0.092567
	$\theta = 0.6827$	0.1115				
Modified Weibull	$\theta = 6.2675$	3.16122	413.969	833.9393	834.1329	0.073875
	$\delta = 6.3551$	3.1869				
	$\alpha = 1.001$	0.0017				

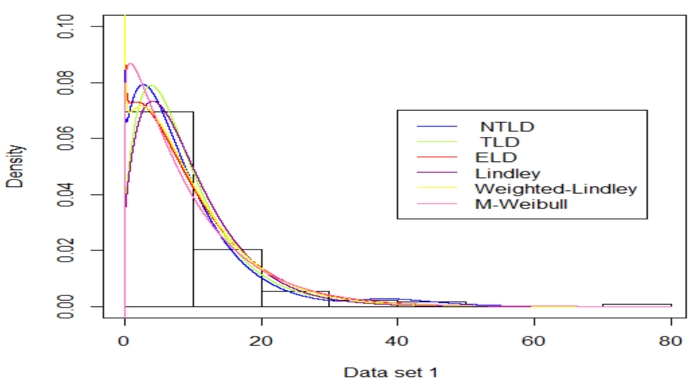

Figure 5: Estimated densities of data set.

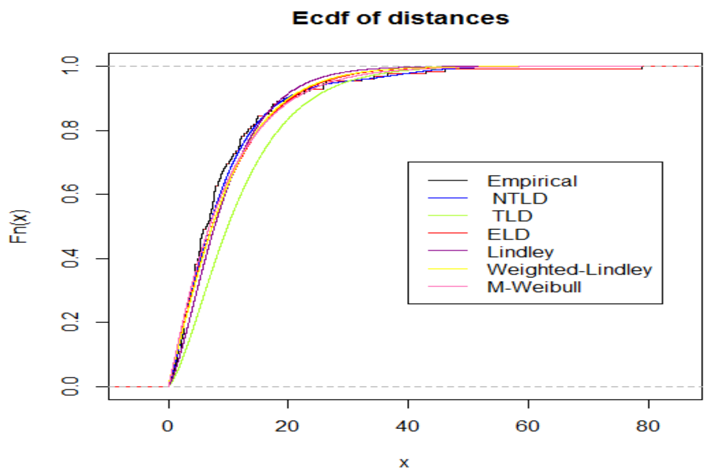

Figure 6: Empirical, fitted NTL, Transmuted Lindely, Exponentiated Lindely, Lindely, weighted Lindely, and Modified Weibull distributions of data set.

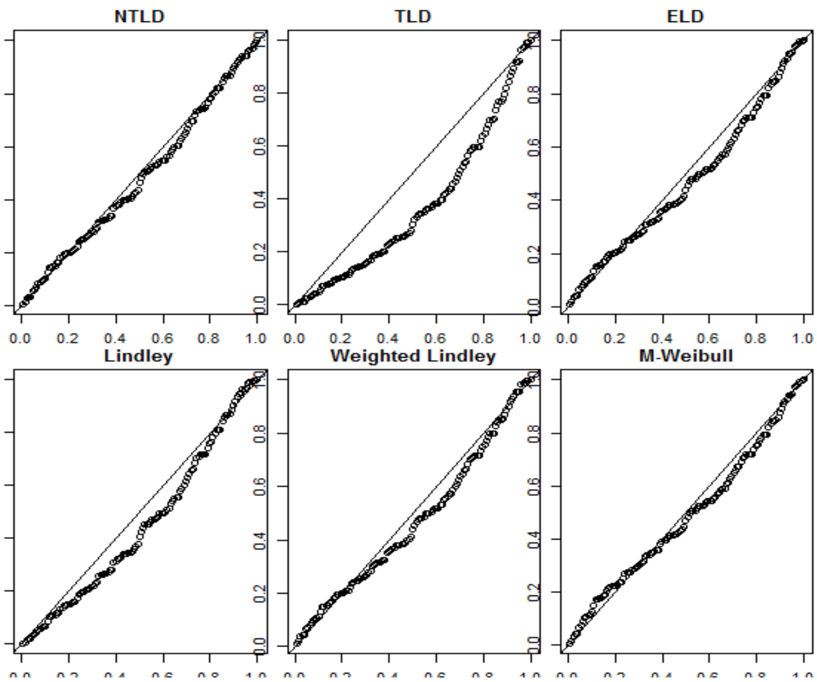

Figure 7: Probability plots for the fits NTL, Transmuted Lindely, Exponentiated Lindely, Lindely, weighted Lindely, and Modified Weibull distributions of data set.

9. Simulation algorithms

In this section we give an algorithm, using R software, to simulate data from the NTL model.

9.1 Inverse CDF method

Since the probability integral transformation cannot be applied explicitly, we, therefore need to follow the following steps for generating a sample of size n from NTL $(\lambda, \theta, \delta, \alpha)$:

Step 1. Set $n, \lambda, \theta, \delta, \alpha$ and initial value x^0.

Step 2. Generate U ~Uniform(0,1).

Step 3. Update x^0 by using the Newton's formula.

$$x^* = x^0 - R(x^0, \Theta),$$

where, $R(x^0, \Theta) = \frac{F_X(x^0,\Theta) - U}{f_X(x^0,\Theta)}$, $F_X(.)$ and $f_X(.)$ are cdf and pdf of NTL distribution, respectively.

Step 4. If $|x^0 - x^*| \leq \epsilon$, (very small, $\varepsilon > 0$ tolerance limit), then store $x = x^*$ as a sample from NTL distribution.

Step 5. If $|x^0 - x^*| > \epsilon$, then, set $x^0 = x^*$ and go to step 3.

Step 6. Repeat steps 3-5, n times for $x_1, x_2, \ldots, x_n$ respectively.

9.2 Inverse CDF method

This subsection explores the behaviors of the proposed estimators in terms of their mean square error on the basis of simulated samples from pdf of NTL with varying sample sizes. We take$\lambda = -0.55$, $\theta = 1, \alpha = 2$arbitrarily and$n = 10(10)100$. The algorithms are coded in R, and the algorithm given in 9.1 has been used for simulation purposes. We calculate MLE estimators of λ, θ, δ and α based on each generated sample. This simulation is repeated 1000 of times, and average estimates with corresponding mean square errors are computed and reported in Table 2.

Table 2. Estimates and mean square errors (in 2-nd row of each cell) of the proposed estimators with varying sample size.

n	*MLE*			
	λ	θ	δ	α
10	-0.5344 0.1370	1.2175 0.1200	3.2465 0.1400	2.7244 1.2127
20	-0.5058 0.0489	1.0022 0.0507	3.0022 0.0507	2.4391 0.3825
30	-0.5406 0.0299	0.9849 0.0407	3.9949 0.0307	2.1573 0.2298
40	-0.5634 0.0253	0.9952 0.0224	3.9952 0.0224	2.1215 0.1590
50	-0.6013 0.0181	0.9954 0.0184	3.9954 0.0184	2.0865 0.1252
60	-0.6121 0.0128	0.9956 0.0148	3.9956 0.0148	2.0804 0.0998
70	-0.6310 0.0119	0.9966 0.0125	3.9966 .0125	2.0711 0.0872
80	-0.6627 0.0100	0.9978 0.0106	3.9978 .0126	2.0553 0.0671
90	-0.6688 0.0089	0.9992 0.0095	3.9992 .0085	2.0511 0.0619
100	-0.6503 0.00161	0.9882 0.0077	3.7882 0.0067	2.0471 0.0445

From Table 2, it can be clearly observed that as sample size increases the mean square error decreases, which proves the consistency of the estimators.

Concluding remarks

There has been a great interest among statisticians and applied researchers in constructing flexible lifetime models to facilitate better modeling of survival data. Consequently, a significant progress has been made towards the generalization of some well-known lifetime models and their successful application to problems in several areas. In this paper, we introduce a new transmuted Lindely distribution obtained using the new generalization technique. We refer to the new model as the NTL distribution and study some of its mathematical and statistical properties. We provide the pdf, the cdf and the hazard rate function of the new model, explicit expressions for the moments. The model parameters are estimated by maximum likelihood and method of moment. The new model is compared with some models and provides consistently better fit than other lifetime models. We hope that the proposed distribution will serve as an alternative model to other models available in the literature.

References

[1] J.C. Ahuja, and S.W. Nash, The generalized Gompertz-Verhulst family of distributions. Sankhya, 29 (1967), 141-161.

[2] G. R. Aryal, & C. P. Tsokos, Transmuted Weibull Distribution: A Generalization of the Weibull Probability Distribution. European Journal of Pure and Applied Mathematics, 4(2) (2011), 89-102.

[3] R. E. Barlow, & F. Proschan, Statistical Theory or Reliability and Life Testing: Probability Models, Tobe with. (1981).

[4] H. W. Block, & T. H. Savits, Burn-in. Statistical Science, 12(1) (1997), 1-19. http://dx.doi.org/10.1214/ss/1029963258

[5] R. C. H Cheng, & N. A. K Amin,. Estimating parameters in continuous univariate distributions with a shifted origin. Journal of the Royal Statistical Society. Series B (Methodological), (1983), 394-403.

[6] I. Elbatal, Exponentiated Modified Weibull Distribution. Economic Quality Control, 26(2) (2011), 189-200. http://dx.doi.org/10.1515/eqc.2011.018

[7] M. A. Greenwich, unimodal hazard rate function and its failure distribution. Statistical Papers, 33(1) (1992), 187-202.
http://dx.doi.org/10.1007/bf02925324

[8] R. D. Gupta, & D. Kundu, Generalized exponential distributions. Australian & New Zealand Journal of Statistics, 41(2) (1999), 173-188.
http://dx.doi.org/10.1111/1467-842x.00072

[9] R. Jiang, P. Ji, & X. Xiao, Aging property of unimodal failure rate models. Reliability Engineering & System Safety, 79(1) (2003), 113-116.
http://dx.doi.org/10.1016/s0951-8320(02)00175-8

[10] M. S. Khan, & R. King, Transmuted Modified Weibull Distribution: A Generalization of the Modified Weibull Probability Distribution. European Journal of Pure and Applied Mathematics, 6(1) (2013), 66-88.

[11] M.M. Mansour, M. A. Enayat, S.H. Mohamed and M. M. Salah, A New Transmuted Additive Weibull Distribution Based on a New Method for Adding a Parameter to a Family of Distributions. International Journal of Applied Mathematical Sciences, (2015).

[12] F. Merovci, Transmuted Lindley Distribution. Int. J. Open Problems Compt. Math., 6(2013), (2), 63-72. http://dx.doi.org/10.12816/0006170

[13] F. Merovci, (2014). Transmuted generalized Rayleigh distribution. Journal of Statistics Applications and Probability, 3(1), 9-20.
http://dx.doi.org/10.12785/jsap/030102

[14] G. S. Mudholkar, & D. K. Srivastava, Exponentiated Weibull family for analyzing bathtub failure-rate data. IEEE Transactions on Reliability, 42(2) (1993), 299-302. http://dx.doi.org/10.1109/24.229504

[15] G. S. Mudholkar, D. K. Srivastava, & M. Freimer, The exponentiated Weibull family: a reanalysis of the bus-motor-failure data. Technometrics, 37(4) (1995). 436-445. http://dx.doi.org/10.2307/1269735

[16] A. M. Sarhan, & M. Zaindin, Modified Weibull distribution. Applied Sciences, 11(1) (2009), 123-136.

[17] W. T. Shaw, I. R. Buckley, The alchemy of probability distributions: beyond Gram-Charlier expansions and a skew-kurtotic-normal distribution from a rank transmutation map. ar Xivpreprint arXiv:0901.0434. (2009).

[18] R. C. R. Team. A Language and Environment for Statistical Computing. R Foundation for Statistical Computing, Vienna, Austria, 2012. ISBN 3-900051-07-0.

[19] M. Xie, & C. D. Lai, Stochastic ageing and dependence for reliability. Springer, New York. (2006). http://dx.doi.org/10.1007/0-387-34232-x

[20] T. Zhang, M. Xie, L. C. Tang, & Ng, S. H. Reliability and modeling of systems integrated with firmware and hardware. International Journal of Reliability, Quality and Safety Engineering, 12(03) (2005), 227-239. http://dx.doi.org/10.1142/s021853930500180x

Received: March 7, 2015; Published: April 3, 2015